Singular Weak Quasitriangular Structures

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1. Quasitriangular Structures

Let K be a field. Throughout, $\otimes = \otimes_K$. Let B be a K-bialgebra and let $B \otimes B$ be the tensor product K-algebra. Let $U(B \otimes B)$ denote the group of units in $B \otimes B$ and let $R \in U(B \otimes B)$.

Definition 1. The pair (B, R) is **almost cocommutative** if

$$\tau(\Delta_B(b)) = R\Delta_B(b)R^{-1} \qquad (1)$$

for all $b \in B$.

If the bialgebra B is cocommutative, then the pair $(B, 1 \otimes 1)$ is almost cocommutative. However, if B is commutative and non-cocommutative, then (B, R) cannot be almost cocommutative for any $R \in U(B \otimes B)$ since in this case (1) reduces to the condition for cocommutativity.

Write $R = \sum_{i=1}^{n} a_i \otimes b_i \in U(B \otimes B)$. Let

$$R^{12} = \sum_{i=1}^{n} a_i \otimes b_i \otimes 1 \in B^{\otimes^3},$$

$$R^{13} = \sum_{i=1}^{n} a_i \otimes 1 \otimes b_i \in B^{\otimes^3},$$

$$R^{23} = \sum_{i=1}^{n} 1 \otimes a_i \otimes b_i \in B^{\otimes^3}.$$

Definition 2. The pair (B, R) is **quasitriangular** if (B, R) is almost cocommutative and the following conditions hold:

$$(\Delta_B \otimes I_B)R = R^{13}R^{23} \tag{2}$$

$$(I_B \otimes \Delta_B)R = R^{13}R^{12} \tag{3}$$

A **quasitriangular structure** is an element $R \in U(B \otimes B)$ so that (B, R) is quasitriangular.

Let (B, R) and (B', R') be quasitriangular bialgebras. Then (B, R), (B', R') are **isomorphic as quasitriangular bialgebras**, written $(B, R) \cong (B', R')$, if there exists a bialgebra isomorphism $\phi : B \to B'$ for which $R' = (\phi \otimes \phi)(R)$. Two quasitriangular structures R, R' on a bialgebra B are **equivalent quasitriangular structures** if $(B, R) \cong (B, R')$ as quasitriangular bialgebras.

Example 3. Suppose that $B = KG^D$ for G finite non-abelian. Then (B, R) cannot be quasitriangular for any $R \in U(B \otimes B)$; B has no quasitriangular structures.

Example 4. Let $n \geq 1$, and let $M_n = \{1, a, a^2, \ldots, a^n\}$ be the monoid with multiplication defined as $a^i a^j = a^{i+j}$ if $i + j \leq n$ and $a^i a^j = a^n$ if $i + j > n$. Let KM_n be the monoid bialgebra with linear dual KM_n^D. By N. Byott [1, slide 14]: $R = 1 \otimes 1$ is the only quasitriangular structure for KM_n and $1 \otimes 1$ is the only quasitriangular stucture for KM_n^D.

Example 5. Let K be a field of characteristic $\neq 2$, let C_2 be the cyclic group of order 2 generated by g and let KC_2 be the group bialgebra. Then there are exactly two non-equivalent quasitriangular structures on KC_2, namely, $R_0 = 1 \otimes 1$ and

$$R_1 = \frac{1}{2}\left(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g\right).$$

Example 6. Let K be a field of characteristic $\neq 2$. Let H be Sweedler's Hopf algebra [4, 1.5.6]:

H is the K-algebra generated by $\{1, g, x, gx\}$ modulo the relations

$$g^2 = 1,\ x^2 = 0,\ xg = -gx,$$

comultiplication $\Delta_H : H \to H \otimes_K H$ is defined by

$$g \mapsto g \otimes g,\ x \mapsto x \otimes 1 + g \otimes x,$$

the counit map $\epsilon_H : H \to K$ is defined as $g \mapsto 1$, $x \mapsto 0$, and the coinverse map $\sigma_H : H \to H$, is given by $g \mapsto g$, $x \mapsto -gx$.

For $a \in K$, let

$$R^{(a)} = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g) + \frac{a}{2}(x \otimes x - x \otimes gx + gx \otimes x + gx \otimes gx)$$

Then $R^{(a)}$ is a quasitriangular structure for H. Moreover, there are an infinite number of non-equivalent quasitriangular structures of the form $R^{(a)}$ for H, cf. [4, 10.1.17], [5].

2. Why We Care

Proposition 7 (Drinfeld [2]). Suppose (B, R) is a quasitriangular bialgebra. Then

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}. \qquad (4)$$

Proof. One has

$$\begin{aligned}
R^{12}R^{13}R^{23} &= R^{12}(\Delta_B \otimes I_B)(R) \quad \text{by (2)} \\
&= (R \otimes 1)(\sum_{i=1}^{n} \Delta_B(a_i) \otimes b_i) \\
&= \sum_{i=1}^{n} R\Delta_B(a_i) \otimes b_i \\
&= \sum_{i=1}^{n} \tau\Delta_B(a_i)R \otimes b_i \quad \text{by (1)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{i=1}^{n} \tau\Delta_B(a_i)\otimes b_i\right)(R\otimes 1)\\
&= (\tau\Delta_B\otimes I_B)\left(\sum_{i=1}^{n} a_i\otimes b_i\right)(R\otimes 1)\\
&= (\tau\Delta_B\otimes I_B)(R)R^{12}\\
&= (\tau\otimes I_B)(\Delta_B\otimes I_B)(R)R^{12}\\
&= (\tau\otimes I_B)(R^{13}R^{23})R^{12} \quad \text{by (2)}\\
&= R^{23}R^{13}R^{12}.
\end{aligned}
$$

□

The equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

is the **quantum Yang-Baxter equation (QYBE)**, [4, Chapter 10].

Proposition 7 says that quasitriangular bialgebras determine solutions to the QYBE.

Also:

Remark 8. Clearly, the QYBE always holds if the bialgebra is commutative. So we really only care in the case B is non-commutative or both non-commutative and non-cocommutative.

Remark 9. To prove Drinfeld's proposition we really didn't need that R is a unit in $B \otimes B$, we only needed the weaker condition: $\tau(\Delta_B(b))R = R\Delta_B(b)$.

Now, suppose (B, R) is a quasitriangular bialgebra of dimension n over K. Let $\{c_1, c_2, \ldots, c_n\}$ be a K-basis for B. Then $\{c_i \otimes c_j \otimes c_k\}$, $1 \leq i, j, k \leq n$, is a K-basis for the n^3-dimensional tensor product algebra $B^{\otimes^3} := B \otimes B \otimes B$.

The matrices in $\mathrm{GL}_{n^3}(K)$ correspond to the collection of invertible linear transformations $B^{\otimes^3} \to B^{\otimes^3}$. Some of the matrices in $\mathrm{GL}_{n^3}(K)$ arise from the elements

$$R^{12} = \sum_i a_i \otimes b_i \otimes 1,$$

$$R^{13} = \sum_i a_i \otimes \otimes 1 \otimes b_i,$$

$$R^{23} = \sum_i 1 \otimes a_i \otimes b_i,$$

as follows.

For each pair $ij = 12, 13, 23$, let

$$R^{ij} : B^{\otimes^3} \to B^{\otimes^3},$$

be the map defined by left multiplication by R^{ij}.

Let μ_{ij} be the transposition maps:

$$\mu_{12} : B^{\otimes^3} \to B^{\otimes^3}, \quad x \otimes y \otimes z \mapsto y \otimes x \otimes z,$$

$$\mu_{13} : B^{\otimes^3} \to B^{\otimes^3}, \quad x \otimes y \otimes z \mapsto z \otimes y \otimes x,$$

$$\mu_{23} : B^{\otimes^3} \to B^{\otimes^3}, \quad x \otimes y \otimes z \mapsto x \otimes z \otimes y.$$

Next, define $R_{ij} : H^{\otimes^3} \to H^{\otimes^3}$ to be the composition of maps $R_{ij} = \mu_{ij} R^{ij}$. Note that R_{12} and R_{23} are invertible K-linear transformations of $H^{\otimes^3}$ which correspond to matrices in $\mathrm{GL}_{n^3}(K)$ (with respect to the K-basis $\{c_i \otimes c_j \otimes c_k\}$).

Proposition 10. *Let K be a field and let (B, R) be a quasitriangular bialgebra of dimension n over K. Then the matrices R_{12}, R_{23} in $GL_{n^3}(K)$ satisfy*

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}. \tag{5}$$

Proof. Use Drinfeld's result. See [6, §4.3]. □

Equation (5) is known as the **braid relation**.

Recall Nigel's Example 4 above:

Proposition 11 (Byott) [1]. *Let $n \geq 1$, and let $M_n = \{1, a, a^2, \ldots, a^n\}$ be the monoid with multiplication defined as $a^i a^j = a^{i+j}$ if $i + j \leq n$ and $a^i a^j = a^n$ if $i + j > n$. Let KM_n be the monoid bialgebra with linear dual KM_n^D. Then $1 \otimes 1$ is the only quasitriangular structure on KM_n^D.*

Proof. Let $B = KM_n^D$. Clearly, $1 \otimes 1 \in U(B \otimes B)$ is a quasitriangular structure on B. Suppose $R \in U(B \otimes B)$ is a quasitriangular structure. Write

$$R = \sum_{a^i, a^j \in M_n} \langle a^i, a^j \rangle e_{a^i} \otimes e_{a^j},$$

for $\langle a^i, a^j \rangle \in K^\times$, $e_{a^i}(a_j) = \delta_{i,j}$.

Thus,

$$(\Delta_B \otimes I_B)R = \sum_{a^r, a^s, a^j \in M} \langle a^r a^s, a^j \rangle e_{a^r} \otimes e_{a^s} \otimes e_{a^j},$$

and

$$R^{13}R^{23} = \left(\sum_{a^i, a^j \in M_n} \langle a^i, a^j \rangle e_{a^i} \otimes 1 \otimes e_{a^j} \right)$$

$$\times \left(\sum_{a^i, a^j \in M_n} \langle a^i, a^j \rangle 1 \otimes e_{a^i} \otimes e_{a^j} \right)$$

$$= \sum_{a^r, a^s, a^j \in M_n} \langle a^r, a^j \rangle \langle a^s, a^j \rangle e_{a^r} \otimes e_{a^s} \otimes e_{a^j}.$$

And so,

$$\langle a^r a^s, a^j \rangle = \langle a^r, a^j \rangle \langle a^s, a^j \rangle,$$

for all $a^r, a^s, a^j \in M_n$.

Now,

$$\langle a^n, a^j \rangle = \langle a^r a^n, a^j \rangle = \langle a^r, a^j \rangle \langle a^n, a^j \rangle,$$

for $a^r, a^j \in M_n$. And so, since $\langle a^n, a^j \rangle \in K^\times$,

$$\langle a^r, a^j \rangle = 1$$

for all $a^r, a^j \in M_n$. It follows that $R = 1 \otimes 1$. $\square$

Remark 12. Similar to Proposition 11, the condition $(I_B \otimes \Delta_B)R = R^{13}R^{12}$ implies

$$\langle a^i, a^r a^s \rangle = \langle a^i, a^r \rangle \langle a^i, a^s \rangle,$$

for all $a^i, a^r, a^s \in M_n$, and so, quasitriangular structures on KM^D correspond to **bimorphisms** $M_n \times M_n \to K^\times$ on M_n.

(Of course, in this case there is only one quasitriangular structure on $B = KM_n^D$, namely the trivial structure $R = 1 \otimes 1$, and consequently, there is exactly one bimorphism on M_n, namely the trivial bimorphism.)

We ask: what happens if we relax the definition of quasitriangular structure?

Suppose we no longer require

$$\tau(\Delta_B(b)) = R\Delta_B(b)R^{-1},$$

(it's weak, as in [1, slide 7]), and we no longer require that R be a unit in $B \otimes B$ (it's singular).

Definition 13. Let B be a K-bialgebra, and let $R \in B \otimes B$. Then R is a **singular weak quasitriangular structure (SWQTS)** on B if

$$(\Delta_B \otimes I_B)R = R^{13}R^{23} \tag{6}$$

$$(I_B \otimes \Delta_B)R = R^{13}R^{12} \tag{7}$$

How do we compute the singular weak quasitriangular structures on B? Of some help may be the following:

Proposition 14 (Drinfeld [3].) *Let* $R = \sum_{i=1}^{n} a_i \otimes b_i$ *be a singular weak quasitriangular structure on* B. *Then*

(i) $(1 \otimes \sum_{i=1}^{n} \epsilon_B(a_i) b_i) R = R$,

(ii) $(\sum_{i=1}^{n} \epsilon_B(b_i) a_i \otimes 1) R = R$.

Proof. See [3], [6, Proposition 4.1.9]. □

(Easier task:) Find all of the singular weak quasitriangular structures on KM_n^D, $n \geq 1$.

Apparently, they will correspond to bimorphisms $M_n \times M_n \to K$ (K, now, and not $K^\times$).

We consider the (modest) cases $n = 1, 2$.

Case I. $n = 1$. Here, $M_1 = \{1, a\}$ with table

	1	a
1	1	a
a	a	a

Let $B = KM_1^D$. Proposition 14 says we should look for structures of the form

$$R = e_1 \otimes e_1 + e_1 \otimes e_a + e_a \otimes e_1 + z e_a \otimes e_a,$$

for $z \in K$.

But then, $(\Delta_B \otimes I_B)R = R^{13}R^{23}$ if and only if $z^2 = z$.

This gives two structures:

$$R_0 = 1 \otimes 1 = e_1 \otimes e_1 + e_1 \otimes e_a + e_a \otimes e_1 + e_a \otimes e_a,$$

which we already knew about, and another:

$$R_1 = e_1 \otimes e_1 + e_1 \otimes e_a + e_a \otimes e_1.$$

We have a non-trivial bimorphism on M_1 defined by

$$\langle 1, 1\rangle = 1,\ \langle 1, a\rangle = 1,\ \langle a, 1\rangle = 1,\ \langle a, a\rangle = 0.$$

Case II. $n = 2$. Here, $M_2 = \{1, a, a^2\}$ with table

	1	a	a^2
1	1	a	a^2
a	a	a^2	a^2
a^2	a^2	a^2	a^2

Let $B = KM_2^D$. Proposition 14 says we should look for structures of the form

$$R = e_1 \otimes e_1 + e_1 \otimes e_a + e_1 \otimes e_{a^2} + e_a \otimes e_1 + we_a \otimes e_a + xe_a \otimes e_{a^2}$$

$$+ e_{a^2} \otimes e_1 + ye_{a^2} \otimes e_a + ze_{a^2} \otimes e_{a^2}$$

$w, x, y, z \in K$.

But then, $(\Delta_B \otimes I_B)R = R^{13}R^{23}$ if and only if

$$\begin{cases} y = w^2 \\ y = y^2 \\ y = wy \\ z = x^2 \\ z = z^2 \\ z = xz \end{cases}$$

Solving this system yields 4 structures on B:

$$R_0 = 1 \otimes 1$$

$$R_1 = e_1 \otimes e_1 + e_1 \otimes e_a + e_1 \otimes e_{a^2} + e_a \otimes e_1 + e_{a^2} \otimes e_1 + e_a \otimes e_{a^2} + e_{a^2} \otimes e_{a^2},$$

$$R_2 = e_1 \otimes e_1 + e_1 \otimes e_a + e_1 \otimes e_{a^2} + e_a \otimes e_1 + e_{a^2} \otimes e_1 + e_a \otimes e_a + e_{a^2} \otimes e_a,$$

$$R_3 = e_1 \otimes e_1 + e_1 \otimes e_a + e_1 \otimes e_{a^2} + e_a \otimes e_1 + e_{a^2} \otimes e_1,$$

and (apparently) 4 bimorphisms $M_2 \times M_2 \to K$ on M_2.

Some of this can be explained by examining the structure of KM_n an a K-algebra.

Proposition 15. *Let $J(KM_n)$ denote the Jacobson radical of KM_n, $n \geq 1$.*

(i) $J(KM_n) = (a - a^n)$, $\dim_K(a - a^n) = n - 1$.

(ii)

$$KM_n \cong Ka \oplus K(1-a) \oplus \bigoplus_{i=1}^{n-1} (a^i - a^n).$$

(iii) $KM_n/J(KM_n) \cong KM_1 \cong K \times K$.

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